A PRE-TEST SHRINKAGE ESTIMATOR OF MEAN OF A NORMAL POPULATION

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1. Introduction

Consider a rondom sample of size n from a normal population with mean μ and variance σ^2, where σ^2 may be unknown. If σ_2 is known, Pradhan [1] proved that a test of H_0 : $\mu=\mu_0$ against H_1 : $\mu=\mu_1$, $(\mu_1>\mu_0)$, which minimizes the sum of the probabilities of two types of error, is given by :

$$\text{Reject } H_0 \text{ if, } \bar{x} \geqslant \frac{\mu_0+\mu_1}{2}, \qquad (1)$$

where $\bar{x}$ is the sample mean. If σ^2 is not known, Singh and Pandey [2] proved that the test (1) still minimizes the sum of the probabilities of two types of error for testing H_0.

Thompson [3] showed that if we have some prior knowledge of the value of μ as μ_0, the shrinkage estimator $k\,\bar{x}+(1-k)\,\mu_0$, $(0\leqslant k \leqslant 1)$, performs better in some region of the parameter space. Now suppose from the familiarity with the experimental material we have two guessed values of μ; that is, we expect either $\mu=\mu_0$ or $\mu=\mu_1$. In this case to decide upon the value of μ, we can perform a preliminary test H_0 according to rule (1). After this we can estimate μ by $k_1\,\bar{x} +(1-k_1)\,\mu_0$, if H_0 is accepted and by $k_2\bar{x}+(1-k_2)\,\mu_1$, if H_0 is rejected, $0\leqslant k_i\leqslant 1$; $i=1, 2$).

To make the problem simpler we have taken $k_1=k_2=k$ and have defined here a preliminary test shrinkage estimator $\hat{\mu}$. We have also discussed its properties and have recommended on the choice of k on the basis of numerical findings.

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2. The Proposed Estimator and its Properties

We suggest the following estimator $\hat{\mu}$ for μ :
We find that,

$$\hat{\mu} \quad \ldots \begin{cases} k\bar{x}+(1-k)\,\mu_0 & \text{if } \bar{x} \leqslant \frac{\mu_0+\mu_1}{2} \\ k\bar{x}+(1-k)\,\mu_1 & \text{if } \bar{x} > \frac{\mu_0+\mu_1}{2} \end{cases} \qquad \ldots(2)$$

We find that,

$$\text{Bias }(\hat{\mu})=E(\hat{\mu})-\mu=(1-k)\,(\mu_1-\mu)+(1-k)\,(\mu_0-\mu_1)\,\phi\,(\mu_0) \qquad \ldots(3)$$

$$\phi(u)=\frac{1}{\sqrt{2\pi}}\,e^{-u^2/2} \text{ and } \phi(t)=\int_{-\infty}^{t}\phi\,(u)du \qquad \ldots(4)$$

Where $u_0=\frac{(\mu_0+\mu_1-2\mu)/\sqrt{n}}{2\sigma}$,

The mean square error (MSE) of $\hat{\mu}$ is

$$MSE\,(\hat{\mu})=E(\hat{\mu}-\mu)^2$$

$$=\frac{\sigma^2}{n}\Big[k^2+(1-k)^2\triangle_1^2+2k\,(1-k)\,(\triangle_1-\triangle_0)\,\phi\,(u_0)$$

$$-(1-k)^2\,(\triangle_1^2-\triangle_0^2)\,\phi(u_0)\Big], \qquad \ldots(5)$$

where $\triangle_1=\frac{\sqrt{n}(\mu_1-\mu)}{\sigma}$ and $\triangle_0=\frac{\sqrt{n}\,(\mu_0-\mu)}{\sigma}$.

2.1 Efficiency of $\hat{\mu}$ w. r. to $\bar{x}$:

We define the efficiency of $\hat{\mu}$ w. r. to $\bar{x}$ as

$$e=\frac{nMSE(\bar{x})}{MSE(\hat{\mu})}=\frac{\sigma^2}{nMSE(\hat{\mu})}.$$

We get,

$$e=[k^2+(1-k)^2\triangle_1^2+2k(1-k)(\triangle_1-\triangle_0)\varphi u_0)$$

$$-(1-k)^2(\triangle_1^2-\triangle_0^2)\phi(u_0)]^{-1} \qquad \ldots\ (6)$$

2.2 THE VALUE OF k FOR WHICH $\hat{\mu}$ HAS MINIMUM MSE :

Differentiating (5) *w.r.* to k we get

$$\frac{\partial\ \text{MLE}\ (\hat{\mu})}{\partial k}=\frac{2\sigma^2}{n}\Big[\,k\Big\{1+\Delta_1^2-2(\Delta_1-\Delta_0)\,\varphi\,(\mu_0)-\Big(\Delta_1^2-\Delta_0^2\Big)\,\Phi\,(u_0)\Big)\Big\}-\Delta_1^2+(\Delta_1-\Delta_0)\,\varphi\,(u_0)+(\Delta_1^2-\Delta_0^2\Big)\,\Phi\,(u_0)\,\Big] \quad ...(7)$$

Equating (7) to zero the optimum value of k for which MSE ($\hat{\mu}$) is minimum is given by

$$k=\frac{\Delta_1^2\,(\Delta_1-\Delta_0)\,\varphi\,(u_0)-(\Delta_1^2-\Delta_0^2\ (\Phi\,(u_0)}{1+\Delta_1^2-2(\Delta_1-\Delta_0)\,\varphi\,(u_0)-\Big(\Delta_1^2-\Delta_0^2\Big)\Phi\,(u_0)}=k_{min}\ \text{(say)} \quad ...(8)$$

2.3 THE RANGE OF VALUES OF k FOR WHICH $\hat{\mu}$ IS MORE EFFICIENT THAN $\bar{x}$

From (6) we see tha $e>1$, if

$$(k-1)\Big[\,k\Big\{1+\Delta_1^2-2(\Delta_1-\Delta_0)\,\varphi\,(u_0)\Big(\Delta_1^2-\Delta_0^2\Big)\,\Phi\,(u_0)\Big\}+1-\Delta_1^2+\Big(\Delta_1^2-\Delta_0^2\Big)\,\Phi\,(u_0)\Big]<0. \quad ...(9)$$

It may be checked, by evaluating

$$\int_{-\infty}^{\frac{\mu_0+\mu_1}{2}}(\bar{x}-\mu_0)^2 f\,(\bar{x})\,d\bar{x}+\int_{\frac{\mu_0+\mu_1}{2}}^{\infty}(\bar{x}-\mu_1)^2 f\,(\bar{x})d\bar{x},$$

that the denominator of expression (8) is positive. If $k<1$, on diving (9) by $k-1$ we get

$$k>k^*, \quad ...(10)$$

where

$$k^*=\frac{\Delta_1^2-1-\Big(\Delta_1^2-\Delta_0^2\Big)\,\Phi(u_0)}{1+\Delta_1^2-2(\Delta_1-\Delta_0)\,\varphi(u_0)-\Big(\Delta_1^2-\Delta_0^2\Big)\Phi(u_0)} \quad ...(11)$$

If $k>1$, on dividing (9) by $k-1$ we get

$$k<k^* \quad ...(12)$$

Combining (10) and (12), we infer that $e>1$, if

$$\left.\begin{array}{ll}\text{either} & k^*<k<1\\ \text{or} & 1<k<k^*\end{array}\right\}. \quad ...(13)$$

The inequality (9) can also be written as

$$k^2\left[1+\Delta_1^2-2(\Delta_1-\Delta_0)\,\varphi(u_0)-\left(\Delta_1^2-\Delta_0^2\right)\Phi(u_0)\right]$$
$$-2k\left[\Delta_1^2-(\Delta_1-\Delta_0)\,\varphi(u_0)-\left(\Delta_1^2-\Delta_0^2\right)\Phi(u_0)\right]$$
$$+\Delta_1^2-1-\left(\Delta_1^2-\Delta_0^2\right)\Phi(u_0)<0. \quad \ldots(14)$$

From (8) and (11) this is equivalent to

$$k^2-2k\,k_{min}+k^*<0. \quad \ldots(15)$$

So, $e>1$,

If $a<k<b$, ...(16)

where $a=k_{min}-\sqrt{k^2_{min}-k^*}$

and $b=k_{min}+\sqrt{k^2_{min}-k^*}$. ...(17)

However, expression (16) holds if $k^2_{min}-k^*$ is positive, that is to say, if the roots of the quadratic equation $k^2-2k\,k_{min}+k^*=0$, are real.

Further it can be noted that,

$$k^*-k_{min}=\frac{(\Delta_1-\Delta_0)\,\varphi(u_0)-1}{1+\Delta_1^2-2(\Delta_1-\Delta_0)\,\varphi(u_0)-\left(\Delta_1^2-\Delta_0^2\right)\Phi(u_0)}$$
$$=\tfrac{1}{2}\,(k^*-1). \quad \ldots(18)$$

So, if $k^*<1$, $k_{min}>k^*$

and if $k^*>1$, $k_{min}<k^*$ ...(19)

Thus if we choose the value of $k=k_{min}$, MSE of $\hat{\mu}$ will be a minimum and will always be less than σ^2/n.

Since we want $0\leqslant k\leqslant 1$, in case $k^*>1$, we take $k=1$. In other situations k^* should be the lower limit of the values of k.

3.—Numerical Findings and Recommendations

Table 1 shows the values of k^* for various Δ_0 and Δ_1. Since we are considering the case $\mu_1>\mu_0$, $\Delta_1>\Delta_0$. Moreover, for any Δ_0 there is only one possible value of Δ_1, *viz.*, $\Delta_0+\dfrac{\mu_1-\mu_0}{\sigma/\sqrt{n}}$.

Since $\dfrac{\mu_1-\mu_0}{\sigma/\sqrt{n}}$ is not known, we consider various values of Δ_1 for a given Δ_0.

... 1

...de of k*

						1	2	3	4	5	6	7	8	9
—5	0.882	.800	.601	.016	—.862	.288	.959	1.215	1.323					
—4	—	.800	.603	.047	—.682	.587	1.226	1.405	1.413					
—3	—	—	.607	.093	—.484	.952	1.503	1.535	1.405					
—2	—	—	—	.103	—.548	1.138	1.659	1.503	1.226					
—1	—	—	—	—	—1.144	0	1.138	.952	.587	.288				
0	—	—	—	—	—	—1.144	—.548	—.484	—.682	—.862	—.956			
1	—	—	—	—	—	—	.103	.093	.047	.016	.004	.002		
2	—	—	—	—	—	—	—	.607	.603	.601	.600	.600	.600	
3	—	—	—	—	—	—	—	—	.800	.800	.800	.800	.800	.800

TABLE 2

Value of *a* and *b*

Δ_0 \ Δ_1	—4	—3	—2	—1	0	1	2	3	4	5	6	7	8
—5	.882 1.000	.800 1.000	.600 1.002	0.16 1.000	—.862 1.000	.288 1.000	.943 1.017	.955 1.221	1.003 1.319				
—4	—	.800 1.000	.601 1.003	.047 .999	—.682 1.000	.586 1.002	1.000 1.226	1.000 1.402	1.002 1.410				
—3	—	—	.609 .997	.093 .999	—.484 1.000	.952 1.000	.998 1.506	1.002 1.532	1.002 1.402				
—2	—	—	—	.103 1.001	—.548 1.000	1.000 1.138	1.002 1.656	.998 1.506	1.000 1.226				
—1	—	—	—	—	—1.144 1.000	0.000 1.000	1.000 1.138	.952 1.000	.586 1.002	0.288 1.000			
1	—	—	—	—	—	—1.144 1.000	—.548 1.000	—.484 1-000	—.682 1.000	—.862 1.000	—.956 1.000		
2	—	—	—	—	—	—	.103 1.001	.093 .999	.047 .999	.016 1.000	.004 1.000	.002 .998	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	—	—		.609 .997	.601 1.003	.600 1.002	.600 1.000	.600 1.000	.600 1.000
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	—		—	.800 1.000	.800 1,000	.800 1.000	.800 1.000	.800 1.000

Table 2 gives the values of a and b for the same Δ_0 and Δ_1 as in Table 1.

From Table 1, we see that:

(*a*) If $\Delta_1=0$

or $\Delta_0=0$

or $(\Delta_0=-1, \Delta_1=1)$

any value of k leads to improvement of $\hat{\mu}$ over $\bar{x}$.

(*b*) If either $0<\Delta_0<2$

or $-2<\Delta_1<0$,

any value of k around .6 will result in improvement of $\hat{\mu}$.

(*c*) If $(\Delta_1=1, \Delta_0=-5)$ or $(\Delta_1=5, \Delta_0=-1)$, k^* is small.

From Table 2, we see that when $k^*<1$, a is approximately equal to k^* and when $k^*>1$, b is approximately equal to k^*. Thus the effective values of k indicated by both the tables are same.

We thus suggest to choose k in the following manner :

(*a*) If μ is expected to lie very near to μ_0 or μ_1 or $\frac{\mu_0+\mu_1}{2}$, any valve of k may be chosen.

(*b*) If μ is not expected to differ from μ_0 or μ_1 by more than $\sigma/\sqrt{n}$, and does not lie between them, we may choose $k>1/10$.

(*c*) If μ does not lie between μ_0 and μ_1 and does not differ from them by more than $\frac{2\sigma}{\sqrt{n}}$, k should be taken around .6.

(*d*) In other situations k should be taken as 1, *i.e.*, $\bar{x}$ should be used to estimate μ.

SUMMARY

In this paper we have suggested a class of pre-test shrinkage estimator of mean of a normal population which is based on two guessed values of mean. The properties of the estimator have been discussed and recommendations on the choice of a particular member have been attempted on the basis of numerical findings.

References

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